THE RIEMANN-HILBERT PROBLEM FOR THE BI-ORTHOGONAL POLYNOMIALS

ANDREI A. KAPAEV

Abstract. For the bi-orthogonal polynomials with the third degree polynomial potential functions, the 3×3 matrix Riemann-Hilbert problem is explicitly constructed. The developed approach admits an extension to the bi-orthogonal polynomials with arbitrary polynomial potentials.

1. Introduction

The classical asymptotic theory of the orthogonal polynomials [1, 2] extended to the polynomials orthogonal w.r.t. the weight $e^{-NV(z)}$ [3, 5, 4, 6] has gained an essential progress after introduction of the Riemann-Hilbert (RH) problem approach [7] and the steepest descent method [8]. A particular implementation of these methods to the orthogonal polynomials on the real line can by found in [9, 10]; a similar study of the orthogonal polynomials on the circle is given in [11].

Further extensions of the notion of the orthogonal polynomials motivated by a number of applications to a random matrix theory, integrable systems, approximation theory and combinatorics include the generalized orthogonal polynomials and the bi-orthogonal polynomials. In the former case, the sequence of the polynomials is orthogonal w.r.t. a sequence of measures [12],

$$\int_{\mathbb{R}} P_n(\lambda)\, d\rho_m(\lambda) = \delta_{nm},$$

while in the latter case, two sequences of the polynomials are orthogonal to each other w.r.t. a two-dimensional measure, see [14, 15, 16],

$$\int_{\mathbb{R}}\int_{\mathbb{R}} P_n(\lambda) Q_m(\xi)\, d\mu(\lambda,\xi) = \delta_{nm}.$$

In the simplest case, the two-dimensional measure has the form of the product $d\mu(\lambda,\xi) = e^{-V(\lambda)-W(\xi)+\lambda\xi}\, d\lambda\, d\xi$ where the polynomials $V(\lambda)$ and $W(\xi)$ are called *potentials*. Integration over ξ in the latter two-fold integral yields a sequence of measures, $d\rho_m(\lambda) = \hat{\phi}_m(\lambda)\, d\lambda = \int_{\mathbb{R}} Q_m(\xi)\, d\mu(\lambda,\xi)$. The functions $\hat{\phi}_m(\lambda)$ here are called the *dual functions*, see [15].

The generalized and bi-orthogonal polynomials are associated to a completely integrable system coming from the t-deformations and the Virasoro constraints and describing typically the reductions of the 2-Toda lattice, see [12, 13, 14, 15, 16]. However, in spite of the algebraic properties of the generalized and bi-orthogonal polynomials are well known, the knowledge of the asymptotic properties of such

This work was partially supported by the EPSRC and by RFBR (grant No 02-01-00268).

The author thanks J. Harnad and M. Bertola for stimulating discussions and valuable comments. The author is also grateful to A. S. Fokas for support and to the staff of DAMTP, University of Cambridge, for the hospitality during his visit.

polynomials is very limited. Basically, this is due to the absence of an adequate formulation of the relevant RH problem. Indeed, in spite of the 2×2 matrix RH problem formulation for the conventional orthogonal polynomials [7] admits a direct 2×2 extension for the generalized orthogonal polynomials [12], the relevant version for the bi-orthogonal polynomials [14] shows its non-local nature.

Fortunately, the study [15] reveals the isomonodromy structure associated to the bi-orthogonal polynomials and hence the principal possibility to formulate an $n \times n$ matrix RH problem which would enjoy the properties similar to those for the 2×2 matrix RH problem for the conventional orthogonal polynomials. In what follows, we construct the 3×3 matrix RH problem for the bi-orthogonal polynomials with the third degree polynomial potentials, as well as the 3×3 RH problem for the relevant dual functions. In spite of its less physical importance, this case provides us with the opportunity to develop the technique in the simplest non-trivial case. Indeed, the bi-orthogonal polynomials for the second degree potentials are reduced to the classical Hermite polynomials [14]. In some more involved case $\deg V(\lambda) > 2$, $\deg W(\xi) = 2$, the bi-orthogonal polynomials can be expressed in terms of the semi-classical orthogonal polynomials related to the 2×2 matrix RH problem studied in [7, 9, 11, 10] and other papers.

We stress that the method presented below can be extended to arbitrary polynomial potentials. We also point out a particular importance of the paper [17] which is useful for construction and justification of the RH problem for the dual functions for the class of the weights with the rational log-derivatives. After the original version of the present paper was posted to the internet, the Montreal group presented their methodology for construction similar RH problem [18]. Their idea for evaluation of the RH problem for the dual functions based on the study of the path integrals coincides with the presented below but, in contrast to our cubic case, they consider the arbitrary polynomial potentials. As to the RH problem for the original (wave) function, the authors of [18] rely on the so-called "duality pairing", while our approach is based on the explicit integral representation for the wave function.

The paper is organized as follows. In Section 2, we recall the matrix differential and difference equations satisfied by the bi-orthogonal polynomials and their dual functions. In Section 3, we construct particular matrix solutions for the differential-difference systems for the dual functions which gives rise to a RH problems for the dual functions (3.3.11)–(3.3.13) and (3.3.15)–(3.3.17). In Section 4, we construct particular solutions of the differential-difference systems for the bi-orthogonal polynomials and the relevant RH problems (4.4.32)–(4.4.34) and (4.4.35)–(4.4.37).

2. Equations for the bi-orthogonal polynomials

Below, we consider the monic polynomials $p_n(\lambda)$, $q_m(\xi)$ satisfying the orthogonality condition

$$\int_{\gamma_1} d\lambda \int_{\gamma_2} d\xi\ p_n(\lambda) q_m(\xi) e^{-V(\lambda)-W(\xi)+t\lambda\xi} = h_n^2 \delta_{n,m}, \tag{2.2.1}$$

where

$$V(\lambda) = \tfrac{1}{3}\lambda^3 + x\lambda, \quad W(\xi) = \tfrac{1}{3}\xi^3 + y\xi, \tag{2.2.2}$$

$x, y, t \in \mathbb{C}$, $t \neq 0$; the contours γ_i, $i = 1, 2$, are the complex linear combinations of the elementary contours,

$$\gamma_i = \sum_{j=0}^{2} g_j^{(i)} \Gamma_j, \quad g_j^{(i)} \in \mathbb{C}, \quad i = 1, 2, \tag{2.2.3}$$

where each Γ_j is the sum of two rays,

$$\Gamma_j = (e^{i\frac{2\pi}{3}(j-2)}\infty, 0] \cup [0, e^{i\frac{2\pi}{3}(j-1)}\infty), \quad j = 0, 1, 2. \tag{2.2.4}$$

Because $\Gamma_0 + \Gamma_1 + \Gamma_2 = 0$, one of the parameters $g_j^{(1)}$ (resp., $g_j^{(2)}$) can be put to zero; one of the nontrivial parameters $g_j^{(i)}$ can be normalized to the unit. Thus the set of contours γ_j and therefore the set of the monic bi-orthogonal polynomials is parameterized by three constant complex parameters[1].

Introduce the wave functions

$$\psi_n(\lambda) = \frac{1}{h_n} p_n(\lambda) e^{-V(\lambda)}, \quad \phi_m(\xi) = \frac{1}{h_m} q_m(\xi) e^{-W(\xi)}, \tag{2.2.5}$$

and define the dual functions $\hat{\psi}_n(\xi)$, $\hat{\phi}_m(\lambda)$ which are the Fourier-Laplace images of the wave functions [15],

$$\hat{\psi}_n(\xi) = \int_{\gamma_1} \psi_n(\lambda) e^{t\lambda\xi}\, d\lambda, \quad \hat{\phi}_m(\lambda) = \int_{\gamma_2} \phi_m(\xi) e^{t\lambda\xi}\, d\xi. \tag{2.2.6}$$

The orthogonality condition (2.2.1) now reads

$$\int_{\gamma_1} d\lambda \int_{\gamma_2} d\xi\, \psi_n(\lambda)\phi_m(\xi) e^{t\lambda\xi} = \int_{\gamma_1} \psi_n(\lambda)\hat{\phi}_m(\lambda)\, d\lambda = \int_{\gamma_2} \hat{\psi}_n(\xi)\phi_m(\xi)\, d\xi = \delta_{nm}\,. \tag{2.2.7}$$

It implies certain relations between the introduced functions (2.2.5), (2.2.6). We refer [15] for the general case and present here the final result for our particular situation:

$$\begin{aligned} &\lambda\psi_n(\lambda) = \sum_{m=n-2}^{n+1} a_{n,m}\psi_m(\lambda), \quad \partial_\lambda \psi_n(\lambda) = -t \sum_{m=n-1}^{n+2} b_{n,m}\psi_m(\lambda), \\ &\partial_x \psi_n(\lambda) = \sum_{m=n}^{n+1} u_{n,m}\psi_m(\lambda), \quad \partial_y \psi_n(\lambda) = -\sum_{m=n-1}^{n} v_{n,m}\psi_m(\lambda), \\ &\partial_t \psi_n(\lambda) = w_n \psi_n(\lambda) - \sum_{m=n-3}^{n-1} A_{n,m}\psi_m(\lambda), \end{aligned} \tag{2.2.8}$$

[1]More general parameterization of the set of contours is introduced in [18], however, in our cubic case, both kinds of parameterizations are equivalent

$$
\begin{aligned}
&\xi\phi_m(\xi) = \sum_{n=m-2}^{m+1} b_{n,m}\phi_n(\xi), \quad \partial_\xi\phi_m(\xi) = -t\sum_{n=m-1}^{m+2} a_{n,m}\phi_n(\xi),\\
&\partial_y\phi_m(\xi) = \sum_{n=m}^{m+1} v_{n,m}\phi_n(\xi), \quad \partial_x\phi_m(\xi) = -\sum_{n=m-1}^{m} u_{n,m}\phi_n(\xi),\\
&\partial_t\phi_m(\xi) = w_m\phi_m(\xi) - \sum_{n=m-3}^{m-1} A_{n,m}\phi_n(\xi),
\end{aligned}
\tag{2.2.9}
$$

where

$$
a_{n,n+1} = b_{n+1,n} = -u_{n,n+1} = -v_{n+1,n} = \frac{h_{n+1}}{h_n}, \quad a_{n+2,n} = b_{n,n+2} = \frac{h_{n+2}}{t\,h_n};
$$

$$
u_{n,n} = \partial_x \ln h_n, \quad v_{n,n} = \partial_y \ln h_n, \quad A_{n,n} = -2w_n = 2\partial_t \ln h_n;
$$

$$
A_{n,m} = \begin{cases} \sum_{k=n-2}^{m+1} a_{n,k}b_{k,m}, & 0 \le n-m \le 3,\\ \sum_{k=m-2}^{n+1} a_{n,k}b_{k,m}, & -3 \le n-m \le 0. \end{cases}
$$

Using the definition (2.2.6) and the equations above, it is straightforward that the dual functions satisfy

$$
\begin{aligned}
&\partial_\xi\hat\psi_n(\xi) = \sum_{m=n-2}^{n+1} t a_{n,m}\hat\psi_m(\xi), \quad \xi\hat\psi_n(\xi) = \sum_{m=n-1}^{n+2} b_{n,m}\hat\psi_m(\xi),\\
&\partial_x\hat\psi_n(\xi) = \sum_{m=n}^{n+1} u_{n,m}\hat\psi_m(\xi), \quad \partial_y\hat\psi_n(\xi) = -\sum_{m=n-1}^{n} v_{n,m}\hat\psi_m(\xi),\\
&\partial_t\hat\psi_n(\xi) = -w_n\hat\psi_n(\xi) + \sum_{m=n+1}^{n+3} A_{n,m}\hat\psi_m(\xi),
\end{aligned}
\tag{2.2.10}
$$

$$
\begin{aligned}
&\partial_\lambda\hat\phi_m(\lambda) = \sum_{n=m-2}^{m+1} t b_{n,m}\hat\phi_n(\lambda), \quad \lambda\hat\phi_m(\lambda) = \sum_{n=m-1}^{m+2} a_{n,m}\hat\phi_n(\lambda),\\
&\partial_y\hat\phi_m(\lambda) = \sum_{n=m}^{m+1} v_{n,m}\hat\phi_n(\lambda), \quad \partial_x\hat\phi_m(\lambda) = -\sum_{n=m-1}^{m} u_{n,m}\hat\phi_n(\lambda),\\
&\partial_t\hat\phi_m(\lambda) = -w_m\hat\phi_m(\lambda) + \sum_{n=m+1}^{m+3} A_{n,m}\hat\phi_n(\lambda).
\end{aligned}
\tag{2.2.11}
$$

The above expressions can be written in the matrix form [15]. Indeed, the 3-vector $\Psi_n(\lambda) = \big(\psi_n(\lambda), \psi_{n-1}(\lambda), \psi_{n-2}(\lambda)\big)^T$ satisfies the system of difference and differential equations,

$$
\begin{aligned}
&\Psi_{n+1}(\lambda) = R_n(\lambda)\Psi_n(\lambda), \quad \frac{\partial\Psi_n}{\partial\lambda}(\lambda) = A_n(\lambda)\Psi_n(\lambda),\\
&\frac{\partial\Psi_n}{\partial x} = U_n(\lambda)\Psi_n, \quad \frac{\partial\Psi_n}{\partial y} = V_n(\lambda)\Psi_n, \quad \frac{\partial\Psi_n}{\partial t} = W_n(\lambda)\Psi_n,
\end{aligned}
\tag{2.2.12}
$$

where

$$
R_n(\lambda) = \begin{pmatrix} \frac{\lambda - a_{n,n}}{a_{n,n+1}} & -\frac{a_{n,n-1}}{a_{n,n+1}} & -\frac{a_{n,n-2}}{a_{n,n+1}}\\ 1 & 0 & 0\\ 0 & 1 & 0 \end{pmatrix},
$$

$$A_n(\lambda) = -D_{n,n+2}R_{n+1}(\lambda)R_n(\lambda) - D_{n,n+1}R_n(\lambda) - D_{n,n} - D_{n,n-1}R_{n-1}^{-1}(\lambda),$$
$$D_{n,m} = t\,\mathrm{diag}\big(b_{n,m}, b_{n-1,m-1}, b_{n-2,m-2}\big),$$
$$U_n(\lambda) = U_{n,n+1}R_n(\lambda) + U_{n,n}, \quad U_{n,m} = \mathrm{diag}\big(u_{n,m}, u_{n-1,m-1}, u_{n-2,m-2}\big),$$
$$V_n(\lambda) = -V_{n,n} - V_{n,n-1}R_{n-1}^{-1}(\lambda), \quad V_{n,m} = \mathrm{diag}\big(v_{n,m}, v_{n-1,m-1}, v_{n-2,m-2}\big),$$

$$W_n(\lambda) = W_{n,n} - \mathcal{A}_{n,n-1}R_{n-1}^{-1}(\lambda) - \mathcal{A}_{n,n-2}R_{n-2}^{-1}(\lambda)R_{n-1}^{-1}(\lambda) - \\ - \mathcal{A}_{n,n-3}R_{n-3}^{-1}(\lambda)R_{n-2}^{-1}(\lambda)R_{n-1}^{-1}(\lambda),$$
$$W_{n,n} = \mathrm{diag}\big(w_n, w_{n-1}, w_{n-2}\big), \quad \mathcal{A}_{n,m} = \mathrm{diag}\big(A_{n,m}, A_{n-1,m-1}, A_{n-2,m-2}\big).$$

Similarly, the 3-vector $\Phi_m(\xi) = \big(\phi_m(\xi), \phi_{m-1}(\xi), \phi_{m-2}(\xi)\big)^T$ satisfies the system

$$\begin{gathered}\Phi_{m+1}(\xi) = Q_m(\xi)\Phi_m(\xi), \quad \frac{\partial \Phi_m}{\partial \xi}(\xi) = B_m(\xi)\Phi_m(\xi), \\ \frac{\partial \Phi_m}{\partial x} = \mathcal{U}_m\Phi_m, \quad \frac{\partial \Phi_m}{\partial y} = \mathcal{V}_m\Phi_m, \quad \frac{\partial \Phi_m}{\partial t} = \mathcal{W}_m\Phi_m,\end{gathered} \tag{2.2.13}$$

where

$$Q_m(\xi) = \begin{pmatrix} \frac{\xi - b_{m,m}}{b_{m+1,m}} & -\frac{b_{m-1,m}}{b_{m+1,m}} & -\frac{b_{m-2,m}}{b_{m+1,m}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$B_m(\xi) = -\mathcal{D}_{m+2,m}Q_{m+1}(\xi)Q_m(\xi) - \mathcal{D}_{m+1,m}Q_m(\xi) - \mathcal{D}_{m,m} - \mathcal{D}_{m-1,m}Q_{m-1}^{-1}(\xi),$$
$$\mathcal{D}_{m,n} = t\,\mathrm{diag}\big(a_{m,n}, a_{m-1,n-1}, a_{m-2,n-2}\big),$$
$$\mathcal{U}_m(\xi) = -U_{m,m} - U_{m-1,m}Q_{m-1}^{-1}(\xi), \quad \mathcal{V}_m(\xi) = V_{m+1,m}Q_m(\xi) + V_{m,m},$$

$$\mathcal{W}_m(\xi) = W_{m,m} - \mathcal{A}_{m-1,m}Q_{m-1}^{-1}(\xi) - \mathcal{A}_{m-2,m}Q_{m-2}^{-1}(\xi)Q_{m-1}^{-1}(\xi) - \\ - \mathcal{A}_{m-3,m}Q_{m-3}^{-1}(\xi)Q_{m-2}^{-1}(\xi)Q_{m-1}^{-1}(\xi).$$

The matrix equations for the 3-vector of dual functions,

$$\hat{\Psi}_n(\xi) = \big(\hat{\psi}_n(\xi), \hat{\psi}_{n-1}(\xi), \hat{\psi}_{n-2}(\xi)\big)^T,$$

are as follows,

$$\begin{gathered}\hat{\Psi}_{n+1}(\xi) = \hat{R}_n(\xi)\hat{\Psi}_n(\xi), \quad \frac{\partial \hat{\Psi}_n}{\partial \xi}(\xi) = \hat{A}_n(\xi)\hat{\Psi}_n(\xi), \\ \frac{\partial \hat{\Psi}_n}{\partial x}(\xi) = \hat{U}_n(\xi)\hat{\Psi}_n(\xi), \quad \frac{\partial \hat{\Psi}_n}{\partial y}(\xi) = \hat{V}_n(\xi)\hat{\Psi}_n(\xi), \quad \frac{\partial \hat{\Psi}_n}{\partial t}(\xi) = \hat{W}_n(\xi)\hat{\Psi}_n(\xi),\end{gathered} \tag{2.2.14}$$

where

$$\hat{R}_n(\xi) = \begin{pmatrix} -\frac{b_{n-1,n}}{b_{n-1,n+1}} & \frac{\xi - b_{n-1,n-1}}{b_{n-1,n+1}} & -\frac{b_{n-1,n-2}}{b_{n-1,n+1}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\hat{A}_n(\xi) = \mathcal{D}_{n,n+1}\hat{R}_n(\xi) + \mathcal{D}_{n,n} + \mathcal{D}_{n,n-1}\hat{R}_{n-1}^{-1}(\xi) + \mathcal{D}_{n,n-2}\hat{R}_{n-2}^{-1}(\xi)\hat{R}_{n-1}^{-1}(\xi),$$
$$\hat{U}_n(\xi) = U_{n,n+1}\hat{R}_n(\xi) + U_{n,n}, \quad \hat{V}_n(\xi) = -V_{n,n} - V_{n,n-1}\hat{R}_{n-1}^{-1}(\xi),$$

$$\hat{W}_n(\xi) = -W_{n,n} + \mathcal{A}_{n,n+3}\hat{R}_{n+2}(\xi)\hat{R}_{n+1}(\xi)\hat{R}_n(\xi) + \\ + \mathcal{A}_{n,n+2}\hat{R}_{n+1}(\xi)\hat{R}_n(\xi) + \mathcal{A}_{n,n+1}\hat{R}_n(\xi).$$

In the very same way, the 3-vector

$$\hat{\Phi}_m(\lambda) = \big(\hat{\phi}_m(\lambda), \hat{\phi}_{m-1}(\lambda), \hat{\phi}_{m-2}(\lambda)\big)^T$$

satisfies the system of equations

$$\begin{gathered}
\hat{\Phi}_{m+1}(\lambda) = \hat{Q}_m(\lambda)\hat{\Phi}_m(\lambda), \quad \frac{\partial \hat{\Phi}_m}{\partial \lambda}(\lambda) = \hat{B}_m(\lambda)\hat{\Phi}_m(\lambda), \\
\frac{\partial \hat{\Phi}_m}{\partial x}(\lambda) = \hat{\mathcal{U}}_m(\lambda)\hat{\Phi}_m(\lambda), \quad \frac{\partial \hat{\Phi}_m}{\partial y}(\lambda) = \hat{\mathcal{V}}_m(\lambda)\hat{\Phi}_m(\lambda), \quad \frac{\partial \hat{\Phi}_m}{\partial t}(\lambda) = \hat{\mathcal{W}}_m(\lambda)\hat{\Phi}_m(\lambda),
\end{gathered} \tag{2.2.15}$$

where

$$\hat{Q}_m(\lambda) = \begin{pmatrix} -\frac{a_{m,m-1}}{a_{m+1,m-1}} & \frac{\lambda - a_{m-1,m-1}}{a_{m+1,m-1}} & -\frac{a_{m-2,m-1}}{a_{m+1,m-1}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\hat{B}_m(\lambda) = D_{m+1,m}\hat{Q}_m(\lambda) + D_{m,m} + D_{m-1,m}\hat{Q}_{m-1}^{-1}(\lambda) + D_{m-2,m}\hat{Q}_{m-2}^{-1}(\lambda)\hat{Q}_{m-1}^{-1}(\lambda),$$

$$\hat{\mathcal{U}}_m(\lambda) = -U_{m,m} - U_{m-1,m}\hat{Q}_{m-1}^{-1}(\lambda), \quad \hat{\mathcal{V}}_m(\lambda) = V_{m+1,m}\hat{Q}_m(\lambda) + V_{m,m},$$

$$\begin{aligned}
\hat{\mathcal{W}}_m(\lambda) = -W_{m,m} + \mathcal{A}_{m+3,m}\hat{Q}_{m+2}(\lambda)\hat{Q}_{m+1}(\lambda)\hat{Q}_m(\lambda) + \\
+ \mathcal{A}_{m+2,m}\hat{Q}_{m+1}(\lambda)\hat{Q}_m(\lambda) + \mathcal{A}_{m+1,m}\hat{Q}_m(\lambda).
\end{aligned}$$

The compatibility conditions of the above equations yield a reduction of the 2-Toda lattice, see [12]. On the other hand, this nonlinear system describes the isomonodromy deformations of the 3×3 matrix differential equations in λ and ξ. The matrix solutions of the systems (2.2.12), (2.2.13), (2.2.14) and (2.2.15) give rise to a matrix RH problems whose data can be found via the asymptotic analysis of the λ- and ξ-equations with the polynomial matrices $A_n(\lambda)$, $B_m(\xi)$ and $\hat{A}_n(\lambda)$, $\hat{B}_m(\xi)$ using the WKB technique, see [19]. It is crucial however, that it is possible to avoid the WKB analysis expanding the basic idea of [7]. In what follows, we construct certain particular solutions of the above systems in terms of the (unknown) bi-orthogonal polynomials using merely the fact of the existence of the linear systems (2.2.12), (2.2.13), (2.2.14), (2.2.15) rather than the systems themselves.

3. The particular solutions of the matrix equations and the Riemann-Hilbert problem for the dual functions

Because of the mentioned above independence of the relevant monodromy data on the deformation parameter t, without loss of generality, we restrict ourselves to $t > 0$. This assumption allows us to simplify our calculations while the final result will be valid for arbitrary $t \in \mathbb{C}\backslash\{0\}$.

Let us introduce the auxiliary functions

$$\hat{\psi}_n^{(j)}(\xi) = \int_{\Gamma_j} \psi_n(\lambda)e^{t\lambda\xi}\, d\lambda, \quad \hat{\phi}_m^{(j)}(\lambda) = \int_{\Gamma_j} \phi_m(\xi)e^{t\lambda\xi}\, d\xi, \tag{3.3.1}$$

where the contours Γ_j, $j = 0, 1, 2$, are defined in (2.2.4). Due to (2.2.6) and (2.2.3), the dual and auxiliary functions are related to each other via

$$\hat{\psi}_n(\xi) = \sum_{j=0}^{2} g_j^{(1)}\hat{\psi}_n^{(j)}(\xi), \quad \hat{\phi}_m(\lambda) = \sum_{j=0}^{2} g_j^{(2)}\hat{\phi}_m^{(j)}(\lambda), \tag{3.3.2}$$

and satisfy the same equations (2.2.10), (2.2.11). Taking into account the orthogonality condition (2.2.7), we find that the functions

$$F_n(\xi) = \frac{e^{W(\xi)}}{2\pi i}\int_{\gamma_2} \frac{\hat\psi_n(\zeta)e^{-W(\zeta)}}{\zeta-\xi}\,d\zeta, \quad G_m(\lambda) = \frac{e^{V(\lambda)}}{2\pi i}\int_{\gamma_1} \frac{\hat\phi_m(\zeta)e^{-V(\zeta)}}{\zeta-\lambda}\,d\zeta, \tag{3.3.3}$$

for $n, m \geq 2$ also satisfy the same equations.

Observing that $\hat\psi_n^{(j)}$, $j = 1, 2$, (respectively, $\hat\phi_m^{(j)}$, $j = 1, 2$) (3.3.1) are combinations of the *independent* Airy functions and their derivatives,

$$\begin{aligned} \hat\psi_n^{(j)}(\xi) &= \frac{1}{h_n} p_n(\partial_\tau)\int_{\Gamma_j} e^{-\frac{1}{3}\lambda^3+\tau\lambda}\,d\lambda\Big|_{\tau=t\xi-x}, \\ \hat\phi_m^{(j)}(\lambda) &= \frac{1}{h_m} q_m(\partial_\tau)\int_{\Gamma_j} e^{-\frac{1}{3}\xi^3+\tau\xi}\,d\xi\Big|_{\tau=t\lambda-y}, \end{aligned} \tag{3.3.4}$$

we construct the particular piece-wise holomorphic solutions of the system of the 3×3 matrix equations (2.2.14) and (2.2.15),

$$\hat\Psi_n(\xi) = \begin{pmatrix} \hat\psi_n^{(1)}(\xi) & \hat\psi_n^{(2)}(\xi) & F_n(\xi) \\ \hat\psi_{n-1}^{(1)}(\xi) & \hat\psi_{n-1}^{(2)}(\xi) & F_{n-1}(\xi) \\ \hat\psi_{n-2}^{(1)}(\xi) & \hat\psi_{n-2}^{(2)}(\xi) & F_{n-2}(\xi) \end{pmatrix}, \quad n \geq 4, \tag{3.3.5}$$

and

$$\hat\Phi_m(\lambda) = \begin{pmatrix} \hat\phi_m^{(1)}(\lambda) & \hat\phi_m^{(2)}(\lambda) & G_m(\lambda) \\ \hat\phi_{m-1}^{(1)}(\lambda) & \hat\phi_{m-1}^{(2)}(\lambda) & G_{m-1}(\lambda) \\ \hat\phi_{m-2}^{(1)}(\lambda) & \hat\phi_{m-2}^{(2)}(\lambda) & G_{m-2}(\lambda) \end{pmatrix}, \quad m \geq 4. \tag{3.3.6}$$

The jump property of the Cauchy integral yields the relations

$$\begin{aligned} F_n^+(\xi) - F_n^-(\xi) &= (g_j^{(2)} - g_{j+1}^{(2)})\hat\psi_n(\xi), \quad \arg\xi = \frac{2\pi}{3}(j-1), \quad j = 0, 1, 2, \\ G_m^+(\lambda) - G_m^-(\lambda) &= (g_j^{(1)} - g_{j+1}^{(1)})\hat\phi_m(\lambda), \quad \arg\lambda = \frac{2\pi}{3}(j-1), \quad j = 0, 1, 2, \end{aligned} \tag{3.3.7}$$

where $g_3^{(i)} \equiv g_0^{(i)}$. Thus we find that the matrix functions $\hat\Psi_n(\xi)$ and $\hat\Phi_m(\lambda)$ have the following jumps across the rays $\ell_j = \{\xi \in \mathbb{C}\colon\ \arg\xi = \frac{2\pi}{3}(j-1)\}$, $j = 0, 1, 2$, oriented towards infinity,

$$\begin{aligned} \hat\Psi_n^+(\xi) &= \hat\Psi_n^-(\xi)\begin{pmatrix} 1 & 0 & (g_j^{(2)} - g_{j+1}^{(2)})(g_1^{(1)} - g_0^{(1)}) \\ 0 & 1 & (g_j^{(2)} - g_{j+1}^{(2)})(g_2^{(1)} - g_0^{(1)}) \\ 0 & 0 & 1 \end{pmatrix}, \quad \xi \in \ell_j, \quad j = 0, 1, 2, \\ \hat\Phi_n^+(\lambda) &= \hat\Phi_n^-(\lambda)\begin{pmatrix} 1 & 0 & (g_j^{(1)} - g_{j+1}^{(1)})(g_1^{(2)} - g_0^{(2)}) \\ 0 & 1 & (g_j^{(1)} - g_{j+1}^{(1)})(g_2^{(2)} - g_0^{(2)}) \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda \in \ell_j, \quad j = 0, 1, 2. \end{aligned} \tag{3.3.8}$$

Using the well known asymptotics of the Airy integrals in the complex domain,

$$\begin{aligned}
&\int_{\Gamma_0} e^{-\frac{1}{3}\lambda^3+\tau\lambda}\,d\lambda = -i\sqrt{\pi}\,\tau^{-1/4}e^{-\frac{2}{3}\tau^{3/2}}\big(1+\mathcal{O}(\tau^{-3/2})\big), \quad \arg\tau\in(-\tfrac{2\pi}{3},\tfrac{2\pi}{3}),\\
&\int_{\Gamma_1} e^{-\frac{1}{3}\lambda^3+\tau\lambda}\,d\lambda = i\sqrt{\pi}\,\tau^{-1/4}e^{-\frac{2}{3}\tau^{3/2}}\big(1+\mathcal{O}(\tau^{-3/2})\big), \quad \arg\tau\in(\tfrac{2\pi}{3},2\pi),\\
&\int_{\Gamma_2} e^{-\frac{1}{3}\lambda^3+\tau\lambda}\,d\lambda = -\sqrt{\pi}\,\tau^{-1/4}e^{\frac{2}{3}\tau^{3/2}}\big(1+\mathcal{O}(\tau^{-3/2})\big), \quad \arg\tau\in(0,\tfrac{4\pi}{3}),
\end{aligned} \tag{3.3.9}$$

the asymptotics of the Cauchy integrals for (3.3.3),

$$\begin{aligned}
F_n(\xi) &= -\frac{h_n}{2\pi i}\xi^{-n-1}e^{W(\xi)}\big(1+\mathcal{O}(\xi^{-1})\big),\\
G_m(\lambda) &= -\frac{h_m}{2\pi i}\lambda^{-m-1}e^{V(\lambda)}\big(1+\mathcal{O}(\lambda^{-1})\big),
\end{aligned} \tag{3.3.10}$$

we construct **the RH problem for the dual functions $\hat{\psi}_n(\xi)$.**

Riemann-Hilbert problem 1. *Find a piece-wise holomorphic 3×3 matrix function $\hat{\Psi}_n^{RH}(\xi)$ with the following properties:*
1. As $\xi\to\infty$,

$$\hat{\Psi}_n^{RH}(\xi) \to \begin{pmatrix}
\frac{\sqrt{\pi}}{h_n}(t\xi)^{\frac{n}{2}-\frac{1}{4}} & \frac{\sqrt{\pi}}{h_n}(-1)^n(t\xi)^{\frac{n}{2}-\frac{1}{4}} & -\frac{h_n}{2\pi i}\xi^{-n-1}\\
\frac{\sqrt{\pi}}{h_{n-1}}(t\xi)^{\frac{n-1}{2}-\frac{1}{4}} & \frac{\sqrt{\pi}}{h_{n-1}}(-1)^{n-1}(t\xi)^{\frac{n-1}{2}-\frac{1}{4}} & -\frac{h_{n-1}}{2\pi i}\xi^{-n}\\
\frac{\sqrt{\pi}}{h_{n-2}}(t\xi)^{\frac{n-2}{2}-\frac{1}{4}} & \frac{\sqrt{\pi}}{h_{n-2}}(-1)^{n-2}(t\xi)^{\frac{n-2}{2}-\frac{1}{4}} & -\frac{h_{n-2}}{2\pi i}\xi^{-n+1}
\end{pmatrix}\times$$
$$\times\begin{pmatrix}
e^{\frac{2}{3}(t\xi)^{3/2}-x(t\xi)^{1/2}} & 0 & \\
0 & e^{-\frac{2}{3}(t\xi)^{3/2}+x(t\xi)^{1/2}} & 0\\
0 & 0 & e^{\frac{1}{3}\xi^3+y\xi}
\end{pmatrix}; \tag{3.3.11}$$

2. Across the rays $\arg\xi = \frac{2\pi}{3}(j-1)$, $j=1,2,3$, oriented towards infinity, $\hat{\Psi}_n^{RH}(\xi)$ has the jumps

$$\hat{\Psi}_n^{RH+}(\xi) = \hat{\Psi}_n^{RH-}(\xi)\hat{S}_j, \quad \arg\xi = \frac{2\pi}{3}(j-1), \tag{3.3.12}$$

where plus and minus indicate the limiting values of $\Psi_n^{RH}(\xi)$ on the jump contour from the left and from the right, respectively, and

$$\hat{S}_1 = \begin{pmatrix}
1 & 0 & (g_1^{(2)}-g_2^{(2)})(g_1^{(1)}-g_2^{(1)})\\
-i & 1 & i(g_1^{(2)}-g_2^{(2)})(g_2^{(1)}-g_0^{(1)})\\
0 & 0 & 1
\end{pmatrix},$$

$$\hat{S}_2 = \begin{pmatrix}
1 & -i & (g_2^{(2)}-g_0^{(2)})(g_1^{(1)}-g_2^{(1)})\\
0 & 1 & i(g_2^{(2)}-g_0^{(2)})(g_1^{(1)}-g_0^{(1)})\\
0 & 0 & 1
\end{pmatrix},$$

$$\hat{S}_3 = \begin{pmatrix}
1 & 0 & (g_0^{(2)}-g_1^{(2)})(g_0^{(1)}-g_2^{(1)})\\
-i & 1 & i(g_0^{(2)}-g_1^{(2)})(g_1^{(1)}-g_0^{(1)})\\
0 & 0 & 1
\end{pmatrix},$$

and across the ray $\arg\xi = -\frac{\pi}{3}$ *oriented towards infinity, the following jump condition holds,*

$$\hat{\Psi}_n^{RH+}(\xi) = \hat{\Psi}_n^{RH-}(\xi)\hat{\Sigma}, \quad \arg\xi = -\frac{\pi}{3}, \quad \hat{\Sigma} = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.3.13}$$

The dual functions $\hat{\psi}_n(\xi)$, $\hat{\psi}_{n-1}(\xi)$ and $\hat{\psi}_{n-2}(\xi)$ form the vector $\hat{\Psi}_n(\xi)$ related to the solution of the RH problem 1 by the following equations,

$$\hat{\Psi}_n(\xi) = \begin{cases} \hat{\Psi}_n^{RH}(\xi)R^{(1)}, & \arg\xi \in (-\frac{\pi}{3}, 0), \\ \hat{\Psi}_n^{RH}(\xi)\hat{S}_1^{-1}R^{(1)}, & \arg\xi \in (0, \frac{2\pi}{3}), \\ \hat{\Psi}_n^{RH}(\xi)\hat{S}_2^{-1}\hat{S}_1^{-1}R^{(1)}, & \arg\xi \in (\frac{2\pi}{3}, \frac{4\pi}{3}), \\ \hat{\Psi}_n^{RH}(\xi)\hat{S}_3^{-1}\hat{S}_2^{-1}\hat{S}_1^{-1}R^{(1)}, & \arg\xi \in (\frac{4\pi}{3}, \frac{5\pi}{3}), \end{cases} \tag{3.3.14}$$

where

$$R^{(1)} = \begin{pmatrix} g_1^{(1)} - g_2^{(1)} \\ i(g_2^{(1)} - g_0^{(1)}) \\ 0 \end{pmatrix}.$$

Riemann-Hilbert problem 2. *Find a piece-wise holomorphic* 3×3 *matrix function* $\hat{\Phi}_m^{RH}(\lambda)$ *with the following properties:*

1. As $\lambda \to \infty$,

$$\hat{\Phi}_m^{RH}(\lambda) \to \begin{pmatrix} \frac{\sqrt{\pi}}{h_m}(t\lambda)^{\frac{m}{2}-\frac{1}{4}} & \frac{\sqrt{\pi}}{h_m}(-1)^m(t\lambda)^{\frac{m}{2}-\frac{1}{4}} & -\frac{h_m}{2\pi i}\lambda^{-m-1} \\ \frac{\sqrt{\pi}}{h_{m-1}}(t\lambda)^{\frac{m-1}{2}-\frac{1}{4}} & \frac{\sqrt{\pi}}{h_{m-1}}(-1)^{m-1}(t\lambda)^{\frac{m-1}{2}-\frac{1}{4}} & -\frac{h_{m-1}}{2\pi i}\lambda^{-m} \\ \frac{\sqrt{\pi}}{h_{m-2}}(t\lambda)^{\frac{m-2}{2}-\frac{1}{4}} & \frac{\sqrt{\pi}}{h_{m-2}}(-1)^{m-2}(t\lambda)^{\frac{m-2}{2}-\frac{1}{4}} & -\frac{h_{m-2}}{2\pi i}\lambda^{-m+1} \end{pmatrix} \times$$
$$\times \begin{pmatrix} e^{\frac{2}{3}(t\lambda)^{3/2}-y(t\lambda)^{1/2}} & 0 \\ 0 & e^{-\frac{2}{3}(t\lambda)^{3/2}+y(t\lambda)^{1/2}} & 0 \\ 0 & 0 & e^{\frac{1}{3}\lambda^3+x\lambda} \end{pmatrix}; \tag{3.3.15}$$

2. Across the rays $\arg\lambda = \frac{2\pi}{3}(j-1)$, $j = 0, 1, 2$, *oriented towards infinity,* $\hat{\Phi}_m^{RH}(\lambda)$ *has the jumps*

$$\hat{\Phi}_m^{RH+}(\lambda) = \hat{\Phi}_m^{RH-}(\lambda)\hat{T}_j, \quad \arg\lambda = \frac{2\pi}{3}(j-1), \tag{3.3.16}$$

where plus and minus indicate the left and right limits of $\Phi_m^{RH}(\lambda)$ *on the jump contour, and*

$$\hat{T}_1 = \begin{pmatrix} 1 & 0 & (g_1^{(1)} - g_2^{(1)})(g_1^{(2)} - g_2^{(2)}) \\ -i & 1 & i(g_1^{(1)} - g_2^{(1)})(g_2^{(2)} - g_0^{(2)}) \\ 0 & 0 & 1 \end{pmatrix},$$

$$\hat{T}_2 = \begin{pmatrix} 1 & -i & (g_2^{(1)} - g_0^{(1)})(g_1^{(2)} - g_2^{(2)}) \\ 0 & 1 & i(g_2^{(1)} - g_0^{(1)})(g_1^{(2)} - g_0^{(2)}) \\ 0 & 0 & 1 \end{pmatrix},$$

$$\hat{T}_3 = \begin{pmatrix} 1 & 0 & (g_0^{(1)} - g_1^{(1)})(g_0^{(2)} - g_2^{(2)}) \\ -i & 1 & i(g_0^{(1)} - g_1^{(1)})(g_1^{(2)} - g_0^{(2)}) \\ 0 & 0 & 1 \end{pmatrix},$$

and across the ray $\arg\lambda = -\frac{\pi}{3}$ *oriented towards infinity, the following jump condition holds,*

$$\hat{\Phi}_m^{RH+}(\lambda) = \hat{\Phi}_n^{RH-}(\lambda)\hat{\Sigma}, \quad \arg\lambda = -\frac{\pi}{3}, \quad \hat{\Sigma} = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.3.17}$$

The dual functions $\hat{\phi}_m(\lambda)$, $\hat{\phi}_{m-1}(\lambda)$ and $\hat{\phi}_{m-2}(\lambda)$ form the vector $\hat{\Phi}_m(\lambda)$ related to the solution of the RH problem 2 by the equations

$$\hat{\Phi}_m(\lambda) = \begin{cases} \hat{\Phi}_m^{RH}(\lambda)R^{(2)}, & \arg\lambda \in (-\frac{\pi}{3}, 0), \\ \hat{\Phi}_m^{RH}(\lambda)\hat{T}_1^{-1}R^{(2)}, & \arg\lambda \in (0, \frac{2\pi}{3}), \\ \hat{\Phi}_m^{RH}(\lambda)\hat{T}_2^{-1}\hat{T}_1^{-1}R^{(2)}, & \arg\lambda \in (\frac{2\pi}{3}, \frac{4\pi}{3}), \\ \hat{\Phi}_m^{RH}(\lambda)\hat{T}_3^{-1}\hat{T}_2^{-1}\hat{T}_1^{-1}R^{(2)}, & \arg\lambda \in (\frac{4\pi}{3}, \frac{5\pi}{3}), \end{cases} \tag{3.3.18}$$

where

$$R^{(2)} = \begin{pmatrix} g_1^{(2)} - g_2^{(2)} \\ i(g_2^{(2)} - g_0^{(2)}) \\ 0 \end{pmatrix}.$$

4. The particular solutions of the matrix equations and the Riemann-Hilbert problem for the wave functions

The method of construction of the Riemann-Hilbert problem for the wave functions $\psi_n(\lambda)$ and $\phi_m(\xi)$ is more involved because of less elementary structure of their integral representations in compare to those for dual functions.

Let $\tilde{\ell}_0^{(j)}$ be an oriented contour connecting a finite point ξ_0 (or λ_0) with infinity within the sector of the exponential decay of the function $\hat{\psi}_n^{(j)}(\xi)$ (3.3.4) (resp., of $\hat{\phi}_m^{(j)}(\lambda)$). Namely, let $\tilde{\ell}_0^{(j)}$ be asymptotic to the ray

$$\tilde{\ell}_0^{(j)} \sim [0, e^{-\frac{2\pi}{3}j}\infty). \tag{4.4.1}$$

Let $\tilde{\Gamma}_j$ be an infinite oriented contour asymptotic to the rays

$$\tilde{\Gamma}_j \sim (e^{i\frac{2\pi}{3}(j-\frac{3}{2})}\infty, 0] \cup [0, e^{i\frac{2\pi}{3}(j-\frac{1}{2})}\infty), \quad j = 0, 1, 2, \tag{4.4.2}$$

located within the sector (4.4.2) and intersecting the ray

$$\ell_j = [0, e^{i\frac{2\pi}{3}(j-1)}\infty), \tag{4.4.3}$$

so that $\tilde{\Gamma}_j \cap \ell_j = \{\xi_j\}$ (resp., $\tilde{\Gamma}_j \cap \ell_j = \{\lambda_j\}$).

The "bricks" which we use to build up the appropriate integral representations are the "inverse" Laplace-Fourier transforms

$$\tilde{\psi}_n^{(j)}(\lambda) = \frac{t}{\pi i}\int_{\tilde{\ell}_0^{(j)}} e^{-t\lambda\xi}\hat{\psi}_n^{(j)}(\xi)\,d\xi, \quad \tilde{F}_n(\lambda) = \frac{t}{\pi i}\int_{\tilde{\Gamma}_j} F_n(\xi)e^{-t\lambda\xi}\,d\xi, \tag{4.4.4}$$

and

$$\tilde{\phi}_m^{(j)}(\xi) = \frac{t}{\pi i}\int_{\tilde{\ell}_0^{(j)}} e^{-t\lambda\xi}\hat{\phi}_m^{(j)}(\lambda)\,d\lambda, \quad \tilde{G}_m(\xi) = \frac{t}{\pi i}\int_{\tilde{\Gamma}_j} G_m(\lambda)e^{-t\lambda\xi}\,d\lambda, \tag{4.4.5}$$

The functions (4.4.4) satisfy the differential equations in λ, x and y in (2.2.8) but not the recursion relation and the differential equation in t because of the

appearance of the inappropriate off-integral terms after integration by parts. The similar observation holds true for (4.4.5). More in details,

$$\lambda\tilde{\psi}_n^{(k)}(\lambda) = \frac{1}{\pi i} e^{-t\lambda\xi_0}\hat{\psi}_n^{(k)}(\xi_0) + \text{appropriate terms}, \tag{4.4.6}$$

$$\begin{aligned} \lambda\tilde{F}_n(\lambda) &= \frac{1}{\pi i} e^{-t\lambda\xi_j}\left(F_n^+(\xi_j) - F_n^-(\xi_j)\right) + \text{appropriate terms} = \\ &= \frac{1}{\pi i} e^{-t\lambda\xi_j}(g_j^{(2)} - g_{j+1}^{(2)})\sum_{k=0}^{2} g_k^{(1)}\hat{\psi}_n^{(k)}(\xi_j) + \text{appropriate terms.} \end{aligned} \tag{4.4.7}$$

In second line of (4.4.7), we have used the jump condition (3.3.7) and the definition (3.3.2). Combining (4.4.6) at $\xi_0 = \xi_j$ and (4.4.7), it is possible to eliminate the off-integral terms and find such a combination $\tilde{F}_n^{(j)}(\lambda)$ of $\tilde{\psi}_n^{(k)}(\lambda)$ and $\tilde{F}_n(\lambda)$ which satisfies the system (2.2.8),

$$\tilde{F}_n^{(j)}(\lambda) = \tilde{F}_n(\lambda) - (g_j^{(2)} - g_{j+1}^{(2)})\sum_{k=0}^{2} g_k^{(1)}\tilde{\psi}_n^{(k)}(\lambda). \tag{4.4.8}$$

Using identity $\sum_j \hat{\psi}^{(j)}(\xi) = 0$, it is possible to eliminate one of the values $\hat{\psi}_n^{(k)}(\xi_j)$ from (4.4.7) and therefore the respective term in (4.4.8). By technical reasons, we prefer to use the latter possibility and introduce the following solutions of (2.2.8):

$$\tilde{F}_n^{(0)}(\lambda) = \tilde{F}_n(\lambda) - (g_0^{(2)} - g_1^{(2)})(g_0^{(1)} - g_1^{(1)})\tilde{\psi}_n^{(0)}(\lambda) - (g_0^{(2)} - g_1^{(2)})(g_2^{(1)} - g_1^{(1)})\tilde{\psi}_n^{(2)}(\lambda), \tag{4.4.9}$$

$$\tilde{F}_n^{(1)}(\lambda) = \tilde{F}_n(\lambda) - (g_1^{(2)} - g_2^{(2)})(g_1^{(1)} - g_0^{(1)})\tilde{\psi}_n^{(1)}(\lambda) - (g_1^{(2)} - g_2^{(2)})(g_2^{(1)} - g_0^{(1)})\tilde{\psi}_n^{(2)}(\lambda), \tag{4.4.10}$$

$$\tilde{F}_n^{(2)}(\lambda) = \tilde{F}_n(\lambda) - (g_2^{(2)} - g_0^{(2)})(g_0^{(1)} - g_2^{(1)})\tilde{\psi}_n^{(0)}(\lambda) - (g_2^{(2)} - g_0^{(2)})(g_1^{(1)} - g_2^{(1)})\tilde{\psi}_n^{(1)}(\lambda), \tag{4.4.11}$$

In the very same way,

$$\tilde{G}_m^{(0)}(\xi) = \tilde{G}_m(\xi) - (g_0^{(1)} - g_1^{(1)})(g_0^{(2)} - g_1^{(2)})\tilde{\phi}_m^{(0)}(\xi) - (g_0^{(1)} - g_1^{(1)})(g_2^{(2)} - g_1^{(2)})\tilde{\phi}_m^{(2)}(\xi), \tag{4.4.12}$$

$$\tilde{G}_m^{(1)}(\xi) = \tilde{G}_m(\xi) - (g_1^{(1)} - g_2^{(1)})(g_1^{(2)} - g_0^{(2)})\tilde{\phi}_m^{(1)}(\xi) - (g_1^{(1)} - g_2^{(1)})(g_2^{(2)} - g_0^{(2)})\tilde{\phi}_m^{(2)}(\xi), \tag{4.4.13}$$

$$\tilde{G}_m^{(2)}(\xi) = \tilde{G}_m(\xi) - (g_2^{(1)} - g_0^{(1)})(g_0^{(2)} - g_2^{(2)})\tilde{\phi}_m^{(0)}(\xi) - (g_2^{(1)} - g_0^{(1)})(g_1^{(2)} - g_2^{(2)})\tilde{\phi}_m^{(1)}(\xi). \tag{4.4.14}$$

Define the 3×3 matrix functions

$$\Psi_n(\lambda) = \begin{pmatrix} \psi_n(\lambda) & \tilde{F}_n^{(0)}(\lambda) & \tilde{F}_n^{(1)}(\lambda) \\ \psi_{n-1}(\lambda) & \tilde{F}_{n-1}^{(0)}(\lambda) & \tilde{F}_{n-1}^{(1)}(\lambda) \\ \psi_{n-2}(\lambda) & \tilde{F}_{n-2}^{(0)}(\lambda) & \tilde{F}_{n-2}^{(1)}(\lambda) \end{pmatrix}, \quad n \geq 4, \tag{4.4.15}$$

and

$$\Phi_m(\xi)=\begin{pmatrix}\phi_m(\xi) & \tilde G_m^{(0)}(\xi) & \tilde G_m^{(1)}(\xi)\\ \phi_{m-1}(\xi) & \tilde G_{m-1}^{(0)}(\xi) & \tilde G_{m-1}^{(1)}(\xi)\\ \phi_{m-2}(\xi) & \tilde G_{m-2}^{(0)}(\xi) & \tilde G_{m-2}^{(1)}(\xi)\end{pmatrix},\quad m\ge 4. \tag{4.4.16}$$

The asymptotics at infinity of $\psi_n(\lambda)$ and $\phi_m(\xi)$ is elementary,

$$\psi_n(\lambda)=\frac{\lambda^n}{h_n}e^{-\frac13\lambda^3-x\lambda}\big(1+\mathcal O(\lambda^{-1})\big),\quad \phi_m(\xi)=\frac{\xi^m}{h_m}e^{-\frac13\xi^3-y\xi}\big(1+\mathcal O(\xi^{-1})\big). \tag{4.4.17}$$

The asymptotics of $\tilde F_n^{(j)}(\lambda)$ and $\tilde G_m^{(j)}(\xi)$ can be found using the conventional steepest descent method,

$$\tilde F_n^{(1)}(\lambda)=\frac{ith_n}{2\pi^{3/2}}(t\lambda)^{-\frac{n+1}{2}-\frac14}e^{-\frac23(t\lambda)^{3/2}+y(t\lambda)^{1/2}}\big(1+\mathcal O(\lambda^{-1/2})\big)+\\ +2(g_1^{(2)}-g_2^{(2)})(g_1^{(1)}-g_0^{(1)})\frac{\lambda^n}{h_n}e^{-\frac13\lambda^3-x\lambda}\big(1+\mathcal O(\lambda^{-1})\big),\quad \arg\lambda\in(-\tfrac{2\pi}{3},0), \tag{4.4.18}$$

$$\tilde F_n^{(1)}(\lambda)=\frac{ith_n}{2\pi^{3/2}}(t\lambda)^{-\frac{n+1}{2}-\frac14}e^{-\frac23(t\lambda)^{3/2}+y(t\lambda)^{1/2}}\big(1+\mathcal O(\lambda^{-1/2})\big)+\\ +2(g_1^{(2)}-g_2^{(2)})(g_2^{(1)}-g_0^{(1)})\frac{\lambda^n}{h_n}e^{-\frac13\lambda^3-x\lambda}\big(1+\mathcal O(\lambda^{-1})\big),\quad \arg\lambda\in(0,\tfrac{2\pi}{3}), \tag{4.4.19}$$

$$\tilde F_n^{(0)}(\lambda)=-\frac{th_n}{2\pi^{3/2}}(-1)^n(t\lambda)^{-\frac{n+1}{2}-\frac14}e^{\frac23(t\lambda)^{3/2}-y(t\lambda)^{1/2}}\big(1+\mathcal O(\lambda^{-1/2})\big)+\\ +2(g_0^{(2)}-g_1^{(2)})(g_2^{(1)}-g_1^{(1)})\frac{\lambda^n}{h_n}e^{-\frac13\lambda^3-x\lambda}\big(1+\mathcal O(\lambda^{-1})\big),\quad \arg\lambda\in(0,\tfrac{2\pi}{3}), \tag{4.4.20}$$

$$\tilde F_n^{(0)}(\lambda)=-\frac{th_n}{2\pi^{3/2}}(-1)^n(t\lambda)^{-\frac{n+1}{2}-\frac14}e^{\frac23(t\lambda)^{3/2}-y(t\lambda)^{1/2}}\big(1+\mathcal O(\lambda^{-1/2})\big)+\\ +2(g_0^{(2)}-g_1^{(2)})(g_0^{(1)}-g_1^{(1)})\frac{\lambda^n}{h_n}e^{-\frac13\lambda^3-x\lambda}\big(1+\mathcal O(\lambda^{-1})\big),\quad \arg\lambda\in(\tfrac{2\pi}{3},\tfrac{4\pi}{3}), \tag{4.4.21}$$

and

$$\tilde F_n^{(2)}(\lambda)=-\frac{ith_n}{2\pi^{3/2}}(t\lambda)^{-\frac{n+1}{2}-\frac14}e^{-\frac23(t\lambda)^{3/2}+y(t\lambda)^{1/2}}\big(1+\mathcal O(\lambda^{-1/2})\big)+\\ +2(g_2^{(2)}-g_0^{(2)})(g_0^{(1)}-g_2^{(1)})\frac{\lambda^n}{h_n}e^{-\frac13\lambda^3-x\lambda}\big(1+\mathcal O(\lambda^{-1})\big),\quad \arg\lambda\in(\tfrac{2\pi}{3},\tfrac{4\pi}{3}), \tag{4.4.22}$$

$$\tilde F_n^{(2)}(\lambda)=-\frac{ith_n}{2\pi^{3/2}}(t\lambda)^{-\frac{n+1}{2}-\frac14}e^{-\frac23(t\lambda)^{3/2}+y(t\lambda)^{1/2}}\big(1+\mathcal O(\lambda^{-1/2})\big)+\\ +2(g_2^{(2)}-g_0^{(2)})(g_1^{(1)}-g_2^{(1)})\frac{\lambda^n}{h_n}e^{-\frac13\lambda^3-x\lambda}\big(1+\mathcal O(\lambda^{-1})\big),\quad \arg\lambda\in(\tfrac{4\pi}{3},2\pi). \tag{4.4.23}$$

For the asymptotics of $\tilde{G}_m(\xi)$, we have

$$\tilde{G}_m^{(1)}(\xi) = \frac{ith_m}{2\pi^{3/2}}(t\xi)^{-\frac{m+1}{2}-\frac{1}{4}}e^{-\frac{2}{3}(t\xi)^{3/2}+x(t\xi)^{1/2}}\big(1+\mathcal{O}(\xi^{-1/2})\big)+ \\ + 2(g_1^{(1)}-g_2^{(1)})(g_1^{(2)}-g_0^{(2)})\frac{\xi^m}{h_m}e^{-\frac{1}{3}\xi^3-y\xi}\big(1+\mathcal{O}(\xi^{-1})\big), \quad \arg\xi\in(-\tfrac{2\pi}{3},0) \tag{4.4.24}$$

$$\tilde{G}_m^{(1)}(\xi) = \frac{ith_m}{2\pi^{3/2}}(t\xi)^{-\frac{m+1}{2}-\frac{1}{4}}e^{-\frac{2}{3}(t\xi)^{3/2}+x(t\xi)^{1/2}}\big(1+\mathcal{O}(\xi^{-1/2})\big)+ \\ + 2(g_1^{(1)}-g_2^{(1)})(g_2^{(2)}-g_0^{(2)})\frac{\xi^m}{h_m}e^{-\frac{1}{3}\xi^3-y\xi}\big(1+\mathcal{O}(\xi^{-1})\big), \quad \arg\xi\in(0,\tfrac{2\pi}{3}) \tag{4.4.25}$$

$$\tilde{G}_m^{(0)}(\xi) = -\frac{th_m}{2\pi^{3/2}}(-1)^m(t\xi)^{-\frac{m+1}{2}-\frac{1}{4}}e^{\frac{2}{3}(t\xi)^{3/2}-x(t\xi)^{1/2}}\big(1+\mathcal{O}(\xi^{-1/2})\big)+ \\ + 2(g_0^{(1)}-g_1^{(1)})(g_2^{(2)}-g_1^{(2)})\frac{\xi^m}{h_m}e^{-\frac{1}{3}\xi^3-y\xi}\big(1+\mathcal{O}(\xi^{-1})\big), \quad \arg\xi\in(0,\tfrac{2\pi}{3}), \tag{4.4.26}$$

$$\tilde{G}_m^{(0)}(\xi) = -\frac{th_m}{2\pi^{3/2}}(-1)^m(t\xi)^{-\frac{m+1}{2}-\frac{1}{4}}e^{\frac{2}{3}(t\xi)^{3/2}-x(t\xi)^{1/2}}\big(1+\mathcal{O}(\xi^{-1/2})\big)+ \\ + 2(g_0^{(1)}-g_1^{(1)})(g_0^{(2)}-g_1^{(2)})\frac{\xi^m}{h_m}e^{-\frac{1}{3}\xi^3-y\xi}\big(1+\mathcal{O}(\xi^{-1})\big), \quad \arg\xi\in(\tfrac{2\pi}{3},\tfrac{4\pi}{3}), \tag{4.4.27}$$

and

$$\tilde{G}_m^{(2)}(\xi) = -\frac{ith_m}{2\pi^{3/2}}(t\xi)^{-\frac{m+1}{2}-\frac{1}{4}}e^{-\frac{2}{3}(t\xi)^{3/2}+x(t\xi)^{1/2}}\big(1+\mathcal{O}(\xi^{-1/2})\big)+ \\ + 2(g_2^{(1)}-g_0^{(1)})(g_0^{(2)}-g_2^{(2)})\frac{\xi^m}{h_m}e^{-\frac{1}{3}\xi^3-y\xi}\big(1+\mathcal{O}(\xi^{-1})\big), \quad \arg\xi\in(\tfrac{2\pi}{3},\tfrac{4\pi}{3}), \tag{4.4.28}$$

$$\tilde{G}_m^{(2)}(\xi) = -\frac{ith_m}{2\pi^{3/2}}(t\xi)^{-\frac{m+1}{2}-\frac{1}{4}}e^{-\frac{2}{3}(t\xi)^{3/2}+x(t\xi)^{1/2}}\big(1+\mathcal{O}(\xi^{-1/2})\big)+ \\ + 2(g_2^{(1)}-g_0^{(1)})(g_1^{(2)}-g_2^{(2)})\frac{\xi^m}{h_m}e^{-\frac{1}{3}\xi^3-y\xi}\big(1+\mathcal{O}(\xi^{-1})\big), \quad \arg\xi\in(\tfrac{4\pi}{3},2\pi). \tag{4.4.29}$$

Using the above asymptotics and the linear constraints for $\tilde{F}_n^{(j)}$, $\tilde{G}_m^{(j)}$,

$$\tilde{F}_n^{(0)}(\lambda)+\tilde{F}_n^{(1)}(\lambda)+\tilde{F}_n^{(2)}(\lambda) = 2g_F\psi_n(\lambda), \\ g_F = g_0^{(2)}(g_2^{(1)}-g_1^{(1)}) + g_1^{(2)}(g_1^{(1)}-g_0^{(1)}) + g_2^{(2)}(g_0^{(1)}-g_2^{(1)}), \tag{4.4.30}$$

and

$$\tilde{G}_m^{(0)}(\xi)+\tilde{G}_m^{(1)}(\xi)+\tilde{G}_m^{(2)}(\xi) = 2g_G\phi_m(\xi), \\ g_G = g_0^{(1)}(g_2^{(2)}-g_1^{(2)}) + g_1^{(1)}(g_1^{(2)}-g_0^{(2)}) + g_2^{(1)}(g_0^{(2)}-g_2^{(2)}), \tag{4.4.31}$$

we find the RH problems for our bi-orthogonal polynomials.

Riemann-Hilbert problem 3. *Find a piece-wise holomorphic* 3×3 *matrix function* $\Psi_n^{RH}(\lambda)$ *with the following properties:*

1. As $\lambda\to\infty$,

$$\Psi_n^{RH}(\lambda)\to\begin{pmatrix}\frac{\lambda^n}{h_n} & -\frac{th_n}{2\pi^{3/2}}(-1)^n(t\lambda)^{-\frac{n+1}{2}-\frac14} & \frac{ith_n}{2\pi^{3/2}}(t\lambda)^{-\frac{n+1}{2}-\frac14}\\ \frac{\lambda^{n-1}}{h_{n-1}} & -\frac{th_{n-1}}{2\pi^{3/2}}(-1)^{n-1}(t\lambda)^{-\frac{n}{2}-\frac14} & \frac{ith_{n-1}}{2\pi^{3/2}}(t\lambda)^{-\frac{n}{2}-\frac14}\\ \frac{\lambda^{n-2}}{h_{n-2}} & -\frac{th_{n-2}}{2\pi^{3/2}}(-1)^{n-2}(t\lambda)^{-\frac{n-1}{2}-\frac14} & \frac{ith_{n-2}}{2\pi^{3/2}}(t\lambda)^{-\frac{n-1}{2}-\frac14}\end{pmatrix}\times$$
$$\times\begin{pmatrix}e^{-\frac13\lambda^3-x\lambda} & 0 & 0\\ 0 & e^{\frac23(t\lambda)^{3/2}-y(t\lambda)^{1/2}} & 0\\ 0 & 0 & e^{-\frac23(t\lambda)^{3/2}+y(t\lambda)^{1/2}}\end{pmatrix} \tag{4.4.32}$$

2. Across the rays $\arg\lambda=\frac{2\pi}{3}(j-1)$, $j=1,2,3$, *oriented towards infinity,* $\Psi_n^{RH}(\lambda)$ *has the jumps*

$$\Psi_n^{RH+}(\lambda)=\Psi_n^{RH-}(\lambda)S_j,\quad \arg\lambda=\frac{2\pi}{3}(j-1), \tag{4.4.33}$$

where plus and minus indicate the limiting values of $\Psi_n^{RH}(\lambda)$ *on the jump contour from the left and from the right, respectively, and*

$$S_1=\begin{pmatrix}1 & 2(g_1^{(2)}-g_0^{(2)})(g_2^{(1)}-g_1^{(1)}) & 2(g_1^{(2)}-g_2^{(2)})(g_1^{(1)}-g_2^{(1)})\\ 0 & 1 & 0\\ 0 & -1 & 1\end{pmatrix},$$

$$S_2=\begin{pmatrix}1 & 2(g_0^{(2)}-g_1^{(2)})(g_2^{(1)}-g_0^{(1)}) & 2(g_0^{(2)}-g_2^{(2)})(g_2^{(1)}-g_0^{(1)})\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix},$$

$$S_3=\begin{pmatrix}1 & 2(g_1^{(2)}-g_2^{(2)})(g_1^{(1)}-g_0^{(1)}) & 2(g_2^{(2)}-g_0^{(2)})(g_1^{(1)}-g_0^{(1)})\\ 0 & 1 & 0\\ 0 & -1 & 1\end{pmatrix},$$

and across the ray $\arg\lambda=-\frac{\pi}{3}$ *oriented towards infinity, the following jump condition holds,*

$$\Psi_n^{RH+}(\lambda)=\Psi_n^{RH-}(\lambda)\Sigma,\quad \arg\lambda=-\frac{\pi}{3},\quad \Sigma=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}. \tag{4.4.34}$$

The wave functions $\psi_n(\lambda)$, $\psi_{n-1}(\lambda)$ and $\psi_{n-2}(\lambda)$ are just entries of the first column of $\Psi_n^{RH}(\lambda)$.

Riemann-Hilbert problem 4. *Find a piece-wise holomorphic* 3×3 *matrix function* $\Phi_m^{RH}(\xi)$ *with the following properties:*

1. As $\xi \to \infty$,

$$\Phi_m^{RH}(\xi) \to \begin{pmatrix} \frac{\xi^m}{h_m} & -\frac{th_m}{2\pi^{3/2}}(-1)^m (t\xi)^{-\frac{m+1}{2}-\frac{1}{4}} & \frac{ith_m}{2\pi^{3/2}}(t\xi)^{-\frac{m+1}{2}-\frac{1}{4}} \\ \frac{\xi^{m-1}}{h_{m-1}} & -\frac{th_{m-1}}{2\pi^{3/2}}(-1)^{m-1}(t\xi)^{-\frac{m}{2}-\frac{1}{4}} & \frac{ith_{m-1}}{2\pi^{3/2}}(t\xi)^{-\frac{m}{2}-\frac{1}{4}} \\ \frac{\xi^{m-2}}{h_{m-2}} & -\frac{th_{m-2}}{2\pi^{3/2}}(-1)^{m-2}(t\xi)^{-\frac{m-1}{2}-\frac{1}{4}} & \frac{ith_{m-2}}{2\pi^{3/2}}(t\xi)^{-\frac{m-1}{2}-\frac{1}{4}} \end{pmatrix} \times$$
$$\times \begin{pmatrix} e^{-\frac{1}{3}\xi^3 - y\xi} & 0 & 0 \\ 0 & e^{\frac{2}{3}(t\xi)^{3/2} - x(t\xi)^{1/2}} & 0 \\ 0 & 0 & e^{-\frac{2}{3}(t\xi)^{3/2} + x(t\xi)^{1/2}} \end{pmatrix} \tag{4.4.35}$$

2. Across the rays $\arg\xi = \frac{2\pi}{3}(j-1)$, $j = 1,2,3$, *oriented towards infinity,* $\Phi_m^{RH}(\xi)$ *has the jumps*

$$\Phi_m^{RH+}(\xi) = \Phi_m^{RH-}(\xi) T_j, \quad \arg\xi = \frac{2\pi}{3}(j-1), \tag{4.4.36}$$

where plus and minus indicate the limiting values of $\Phi_m^{RH}(\xi)$ *on the jump contour from the left and from the right, respectively, and*

$$T_1 = \begin{pmatrix} 1 & 2(g_1^{(1)} - g_0^{(1)})(g_2^{(2)} - g_1^{(2)}) & 2(g_1^{(1)} - g_2^{(1)})(g_1^{(2)} - g_2^{(2)}) \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 1 & 2(g_0^{(1)} - g_1^{(1)})(g_2^{(2)} - g_0^{(2)}) & 2(g_0^{(1)} - g_2^{(1)})(g_2^{(2)} - g_0^{(2)}) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 1 & 2(g_1^{(1)} - g_2^{(1)})(g_1^{(2)} - g_0^{(2)}) & 2(g_2^{(1)} - g_0^{(1)})(g_1^{(2)} - g_0^{(2)}) \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

and across the ray $\arg\lambda = -\frac{\pi}{3}$ *oriented towards infinity, the following jump condition holds,*

$$\Phi_m^{RH+}(\xi) = \Phi_m^{RH-}(\xi)\Sigma, \quad \arg\xi = -\frac{\pi}{3}, \quad \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.4.37}$$

The wave functions $\phi_n(\lambda)$, $\phi_{n-1}(\lambda)$ and $\phi_{n-2}(\lambda)$ are just entries of the first column of $\Phi_n^{RH}(\xi)$.

References

[1] M. Plancherel and W. Rotach, Sur les valeurs asymptotiques des poynomes d'Hermite $H_n(x) = (-1)^n e^{x^2/2} d^n (e^{-x^2/2})/dx^n$, *Comment. Math. Helvet.* **1** (1929) 227–254

[2] G. Szegö, *Orthogonal Polynomials*, AMS, Providence, Rhode Island, 1939

[3] G. Freud, On the coefficients in the recursion formulae of orthogonal polynomials, *Proc. Roy. Irish Acad. Sect. A* **76** (1976) 1–6

[4] A. P. Magnus, Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials, *J. Comput. Appl. Math.* **57** (1995) 215–237

[5] P. Nevai, Géza Freud, orthogonal polynomials and Christoffel functions. A case study, *J. Approx. Theory* **48** (1986) 3–167

[6] R.-C. Sheen, Plancherel-Rotach-type asymptotics for orthogonal polynomials associated with $\exp(-x^6/6)$, J. Approx. Theory **50** (1987) 232–293

[7] A. S. Fokas, A. R. Its and A. V. Kitaev, Isomonodromic approach in the theory of two-dimensional quantum gravity, *Uspekhi Mat. Nauk* **45** (1990) 135–136

______, Discrete Painlevé equations and their appearance in quantum gravity, *Comm. Math. Phys.* **142** (1991) 313–344

______, The isomonodromy approach to matrix models in 2D quantum gravity, *Comm. Math. Phys.* **147** (1992) 395–430

[8] P. A. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, *Ann. of Math.* **137** (1993) 295–368

[9] P. M. Bleher and A. R. Its, Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model, *Ann. Math.* **150** (1999) 185–266

______, Double scaling limit in the random matrix model: Riemann-Hilbert approach, math-ph/0201003

[10] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides and Z. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Commun. Pure Appl. Math.* **52** (1999) 1335–1425;

______, Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Commun. Pure Appl. Math.* **52** (1999) 1491–1552

[11] J. Baik, P. Deift and K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, *J. Am. Math. Soc.* **12** (1999) 1119–1178, MathCO/9810105

[12] M. Adler and P. van Moerbeke, Generalized orthogonal polynomials, discrete KP and Riemann-Hilbert problems, nlinSI/0009002

[13] P. van Moerbeke, *Integrable lattices: random matrices and random permutations*, MSRI Publications #40, Cambridge University Press, 2001, math.CO/0010135

[14] N. M. Ercolani and K. T.-R. McLaughlin, Asymptotics and integrable structures for biorthogonal polynomials associated to a random two-matrix model, *Physica D* **152-153** (2001) 232–268

[15] M. Bertola, B. Eynard and J. Harnad, Duality, biorthogonal polynomials and multi-matrix models, nlin.SI/0108049

[16] P. Zinn-Justin, HCIZ integral and 2D Toda lattice hierarchy, math-ph/0202045

[17] M. Bertola, Bilinear semi-classical moment functionals and their integral representation, *Preprint* CRM-2842 (2002), math.CA/0205160

[18] M. Bertola, B. Eynard and J. Harnad, Differential systems appearing in 2-matrix models and the associated Riemann-Hilbert problem, nlin.SI/0208002

[19] A. R. Its and V. Yu. Novokshenov, The Isomonodromy Deformation Method in the Theory of Painlevé Equations, *Lect. Notes Math.* **1191** Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1986

St Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences, Fontanka 27, St Petersburg, 191011, Russia

Current address: Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver st., Cambridge, CB3 9EW, England

E-mail address: `kapaev@pdmi.ras.ru, A.Kapaev@damtp.cam.ac.uk`